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ON THE BOUNDED OF VERTICES FOR THE REFLEXIVE POLYTOPE

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ABSTRACT

In this work the basic concept of the reflexive polytope with Earhart series for it are given. Two known inequalities on h-vector for the reflexive poytope are presented. By using them we give a complete classification of possible h-vector of integral reflexive polytope with $\sum_{i=0}^{d} h_i \leq 3$, the bounded of the number of vertices for the reflexive polytope are also proven.

KEYWORDS: On the Bounded of Vertices for the Reflexive Polytope

INTRODUCTION

Let P$\subseteq R^d$ be a reflexive polytope, which is any vertex of P is an integral that is the coordinate of the vertices belong to the set of integers Z, and contain only the origin in its interior and the dual P^* of P is also integral. The dimension of P is d, ∂P denoted the boundary of P.

The reflexive polytope P were originally defined with theoretical physics applications in mind in string theory, reflexive polytope and the associated toric varieties play crucial role in the most quantitatively predictive form of the mirror symmetry, [7].

Two theoretical physicists Maximillian Kreuzer and Harald Skark, worked out a dilated description of reflexive polytope in the late 1990s. Their motivation came from string theory: using polytopes, physicists were able to be construct in geometry space that could model extra dimension of our universe, [5].

Duality in convex polyhedra is a very interesting notion not only in the theory of polyhedral but also in polyhedral computation. Duality implies that two basic representation conversions between V-representation and H-representation of a polyhedron are essentially the same thing. Yet, in order to convert one to the other is sometimes tricky because there are certain assumptions under which any specific conversion can work,[6].

Let LP=#(tP$\cap Z^d$) denote the number of integral points in an integral dilated tp of P by factor of t$\in Z_{\geq O}$ where tP={tx:x$\in P$}, [8].

The France mathematical Eugene Eharhart in 1967 found a way to prove that the function count the number of integral points in a polytope that lie inside tP which is a polynomial in t [8], it is denoted by L(p,t)=$\sum_{i=0}^{d} c_i t^i$

In 1971, I. G. Macdonald proved the following reciprocity theorem, which had been conjectured (and proved for several special cases) by Ehrhart, where P° denoted the interior of P

$$(\mathrm{P}, -\mathrm{t}) = (-1)^{\mathrm{d}} \mathrm{L}(\mathrm{P}^\circ, \mathrm{t})$$

Equivalently, the companying generating function (the Ehrhart series of P) evaluates to a rational function

$\mathrm{Ehrp}(Z)=\sum_{i\geq 0} L(P,t)Z^i=\frac{h_0+h_1Z+\cdots+h_dZ^d}{(1-Z)^{d+1}}$

The coefficients h_i are non-negative, the sequence $(h_0, h_1, \ldots, h_d)$ is known as h-vector of p, [3].

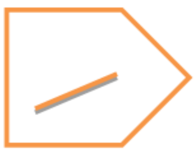
For an elementary proof of this and other relevant results see [9].

1. BASIC CONCEPT

In this section some basic definitions of convex bounded polyhedron and its representation with some examples are given.

Definition (1), [14]

A set S $\subseteq R^d$ is said convex if the entire line segment between any two vectors in S is contained in S. that is S is convex if and only if $\{\lambda x_1 + (1-\lambda)x_2 : 0 \leq \lambda \leq 1\} \subseteq$ S for every $x_1, x_2 \in S$.



Convex

Not Convex

Definition (2), [11]

The set of solutions

$$P=\left\{(x_1,\ldots x_d)\in R^d \,\middle|\, \begin{array}{c} a_{11}x_1+\cdots+a_{1n}x_d \leq b_1 \\ . \\ . \\ . \\ a_{m1}x_1+\cdots+a_{md}x_d \leq b_m \end{array}\right\}$$

To the system

$$\begin{array}{c} a_{11}x_1+\cdots+a_{1n}x_d \leq b_1 \\ . \\ . \\ . \\ a_{m1}x_1+\cdots+a_{md}x_d \leq b_m \end{array}$$

Of a finitely many linear inequalitieshere (a_{ij} *and* b_j are real numbers) is called a polyhedron

Definition (3),[16]

For A polyhedronP$\subseteq R^d$ is bounded such that there exist $\omega \in R^d$ such that $\|X\| \leq \omega$ where X$\in P$then P is polytope

2. REPRESENTATION OF POLYHEDRON, [4]

There are two representations for a polyhedronnamely H-representation and V-representation, when P={X$\in R^d$:

AX$\leq b\}$ for some real matrix A and a vector b the pair (A,b) is called a half space represented or simply H-representation of P. when P is convex hull of V then it called vertex representation or simply V-representation of P.

Definition (4),[13]

A convex polytope P$\subseteq R^d$ is called integral if all vertices of P belong to Z^d.

Definition (5),[12]

When P is a polytope in R^d containing the origin in the interior in tP a convex set P^* in R^d is defined by $\mathrm{P}^* \coloneqq \{\mathrm{u} \in \mathrm{R}^{\mathrm{d}}: \langle \mathrm{u}, \mathrm{x}\rangle \leq 1 \forall \mathrm{x} \in \mathrm{P}\}$ this polytope is called dual of P and $P^{**} = P$.

Definition (6),[2]

Let P$\subseteq \mathrm{R}^{\mathrm{d}}$ is a convex d-dimensional polytope with vertices in Z^d(integral polytope) with $0 \in \mathrm{int}\mathrm{P}$ such that the dual P^* is also integral polytope then the polytope P is reflexive.

Definition (7),[1]

The Eharhart Polynomial L(P,t) = #(tP$\cap Z^d$) of an integral polytope P is given by the number of integral points contained in the dilation of the polytope , is that tP with t$\in Z$ the degree of L(P,t) is the dimension of the polytope.

Definition (8),[9]

The vector of d-polytope P is the sequence $\mathrm{h}_0(\mathrm{p}), \mathrm{h}_1(\mathrm{p}), \ldots \mathrm{h}_{\mathrm{d}}(\mathrm{p})$ where$\mathrm{h}_{\mathrm{d}} \neq 0$and $\mathrm{h}_0 = 1, \mathrm{h}_1 = \#(\mathrm{P} \cap \mathrm{Z}^{\mathrm{d}}) - (\mathrm{d}+1), \mathrm{h}_{\mathrm{d}} = \#(\mathrm{P}^{\circ} \cap \mathrm{Z}^{\mathrm{d}})$ moreover each h_{i} is non-negative is called h-vector

3. THE ERHART SERIES AND H-VECTOR FOR THE REFLEXIVE POLYTOPE

In this section we discuss h-vector of reflexive polytope and shows two known-inequalities on h-vector for reflexive polytope. Our main goal is to give the bounds of the number of vertices of reflexive polytope in dimension two by using h-vector and shows the inequalities associated with the reflexive polytope P.

3.1. Compution of h-vector of integral polytope, [10]

Before we explain our goal we try to show the combinatorial technique to compute the h-vector of an integral polytope.

Let P$\subseteq \mathrm{R}^{\mathrm{d}}$be an integral polytope with the vertices $\mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{\mathrm{d}}$we set

S $=\{ \sum_{\mathrm{i}=0}^{\mathrm{d}} \mathrm{r}_{\mathrm{i}} (\mathrm{v}_{\mathrm{i}}, 1) \in \mathrm{R}^{\mathrm{d}+1} : 0 \leq \mathrm{r}_{\mathrm{i}} < 1\} \cap \mathrm{Z}^{\mathrm{d}+1}$ and

$\mathrm{S}^*= \{ \sum_{\mathrm{i}=0}^{\mathrm{d}} \mathrm{r}_{\mathrm{i}} (\mathrm{v}_{\mathrm{i}}, 1) \in \mathrm{R}^{\mathrm{d}+1} : 0 < \mathrm{r}_{\mathrm{i}} \leq 1\} \cap \mathrm{Z}^{\mathrm{d}+1}$.

The degree of an integral points $(\alpha, \mathrm{t}) \in \mathrm{S}, \left((\alpha, \mathrm{t}) \in \mathrm{S}^*\right)$ with deg $(\alpha, \mathrm{t}) = \mathrm{t}$ where $\alpha \in \mathrm{Z}^{\mathrm{d}}$ and t$\in \mathrm{Z}_{\geq 0}$. let $\mathrm{h}_{\mathrm{i}} = |\{\alpha \in \mathrm{h}: \deg \alpha = \mathrm{i}\}|$ and $\mathrm{h}_{\mathrm{i}}^* = |\{\alpha \in \mathrm{h}^*: \deg \alpha = \mathrm{i}\}|$

Then we have

Lemma (1)

Work with the notation as above. Then

$\sum_{t=0}^{\infty} L(p,t)\lambda^{t} = \frac{h_0+h_1\lambda+\cdots+h_d\lambda^{d}}{(1-\lambda)^{d+1}}$;

$\sum_{t=0}^{\infty} L(P^{*},t)\lambda = \frac{h_1^{*}\lambda+\cdots h_{d+1}^{*}\lambda}{(1-\lambda)^{d+1}}$;

$h_i^{*} = h_{d+1-i}$ for $1\leq i \leq d+1$.

Lemma (2)

Suppose that $(h_0, h_1, \dots, h_d)$ is the h-vector of an integral convex polytope of dimension d. then there exists an integral convex poytope of dimension d+1 whose h-vector is $(h_0, h_1, \dots h_d, 0)$.

Lemma (3),[15]

Let P be convex d-polytope in R^{d} with integer vertices let L(P,t) denote its ehrhart polynomial and write $\sum_{t\geq 0} L(p,t)Z^{i} = \frac{h_0+h_1 z+\cdots h_s z^{s}}{(1-z)^{d+1}}$

Where $h_s \neq 0$(since L(p,t) is a polynomial for all t we have s$\leq$ d) then

$h_0+h_1+\cdots h_i \leq h_s+h_{s-1}+\cdots h_{s-i}$

For all $0 \leq i \leq s$

In [10] presented two well-known inequalities on h-vector, used them to give the complete classification of the possible h-vector of an integral convex polytope with $\sum_{i=0}^{d} h_i \leq 3$. Let s=max $\{i: h_i \neq 0\}$ Stanly in [R.P.Stanly 1991] shows the inequalities:

$$h_0+h_1+\cdots+h_i \leq h_s+h_{s-1}+\cdots h_{s-i}\ 0 \leq i \leq \frac{s}{2}$$

By using the theory of Cohen Macaualy rings on the other side the inequalities

$$h_d+h_{d-1}+\cdots h_{d-i} \leq h_1+h_2+\cdots h_{i+1}\ \ 0 \leq i \leq \frac{d-1}{2}$$

By using the above inequalities together we can give the bound on the number of vertices of the reflexive polytope.

Proposition (1)

Given a finite sequence $(h_0, h_1, \dots, h_d)$ of non-negative integers, with $h_0 = 1$ which satisfies $\sum_{i=0}^{d} h_i \leq 3$ there exist a reflexive polytope $P\subset R^{d}$ of dimension d=1, 2, 3 if the number of integral point of P is less than or equal four

Proof

We have $\sum_{i=0}^{d} h_i \leq 3$ which means that $h_0+h_1+h_2+h_3 \leq 3$ when d=3, $h_0 = 1$, $h_1 = \#(P\cap Z^{d})$ –(d+1) and $h_3 = \#\left(P^{\circ}\cap Z^{d}\right)$we get

$$\sum_{i=0}^{d} h_i = h_0+h_1+h_2+h_3 \leq 3$$

$$= 1 + \#(P\cap Z^{3}) - d - 1 + h_2 + \#(P^{\circ}\cap Z^{3}) \leq 3$$

$$= \#(P\cap Z^{3}) - h_2 \leq 3 + (d-1)$$

$= \#(P \cap Z^3) \leq 5 - h_2$ Where $h_2 \leq 1$ ■

Proposition (2)

For a reflexive polytope $P \subseteq R^2$. Given a finite sequence $(h_0, h_1, \dots, h_d)$ of nonnegative integers where $h_0 = 1$ which satisfies $\sum_{i=0}^{2} h_i \leq 3$, if the number of integral points is $4 \leq n \leq 9$ then $\sum_{i=1}^{2} n - h_i \leq 3$.

Proof

The reflexive polytope p which is contains only the origin in its interior, $h_0 = 1$, then

$\sum_{i=0}^{d} n - h_i \leq 3$

$= n - (h_0 + h_1 + h_{2)} \leq 3$

$n - (1 + \#(P \cap Z^2) - (d+1) \leq 3$

$= n - (1 + n - d) \leq 3$

this mean $d \leq 2$ ■

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